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A decomposition theorem for mixed classical and Ising spin systems†

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Abstract. Rigorous results are presented for d-dimensional lattices decorated with ν-dimensional classical spins, whose sites are occupied by Ising spins. It is shown that for suitable interactions the effect of such decorations is to generate temperature-dependent interactions between the Ising spins. It is also shown that for a system composed of two Ising–classical spins systems, which interact via a single bond, the canonical average of an arbitrary function of the spins (Ising and/or classical) in one of the systems does not depend on the interaction parameters contained in the other.

1. Introduction

Let us consider a d-dimensional lattice with an Ising spin $\boldsymbol{\sigma}_j$ on each site and a classical spin $\boldsymbol{S}_{ij}$ on each bond connecting the two sites i, j. The classical spin is a unit vector with ν components and $\boldsymbol{\sigma}_j = \sigma_j \hat{u}_j^\nu$ where $\hat{u}_j^\nu$ is the unit vector in the ν-direction and $\sigma_j = \pm 1$. We also consider that the Ising spins on the two particular sites i, j interact through the classical spins via the interaction

$$-J_{ij}(\boldsymbol{\sigma}_i \cdot \boldsymbol{S}_{ij} + \boldsymbol{\sigma}_j \cdot \boldsymbol{S}_{ij}). \tag{1}$$

This interaction for classical planar spins ($\nu = 2$) has been proposed by Falk (1980) and it corresponds to the decoration of the bonds. This type of decoration with spins of different kinds has been considered for Ising spins only (different s) and was introduced by Fisher (1959) among others. Falk (1980) considered a model for $\nu = 2$ in one- and two-dimensional lattices, where the interactions between all nearest-neighbour pairs are given by equation (1) and solved it exactly. This model is still solvable for arbitrary ν in both lattices and the solution has been recently obtained (Gonçalves 1982).

In this paper we will consider rather general mixed spin systems in d-dimension lattices which interact via a single bond through an interaction of the form shown in equation (1). The main purpose is to prove that under these conditions the systems behave as independent systems. This proof is based on results obtained by Falk (1974, 1975) for systems containing Ising spins only.

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The total Hamiltonian of the system, writing explicitly the interaction between the pair (i, j) can be written in the form

$$H = \sum_{(k,l)}{}' f(J_{kl}, \boldsymbol{\sigma}_k, \boldsymbol{\sigma}_l, \boldsymbol{S}_{kl}) - J_{ij}(\sigma_i S_{ij}^{\nu} + \sigma_j S_{ij}^{\nu}) \tag{2}$$

where the restricted sum does not include the pair (i, j) and S_{ij}^{ν} is the ν component of the vector $\boldsymbol{S}_{ij}$.

The partition function is then given by

$$Z = \sum_{\{\sigma\}} \int \prod_{(k,l)} \mathrm{d}\Omega_{kl}\, \mathrm{d}\Omega_{ij} \exp\Big(-\beta \sum_{(k,l)}{}' f(J_{kl}, \boldsymbol{\sigma}_k, \boldsymbol{\sigma}_l, \boldsymbol{S}_{kl}) + \beta J_{ij}(\sigma_i + \sigma_j) S_{ij}^{\nu}\Big) \tag{3}$$

where $\beta = 1/K_\mathrm{B}T$ and $\mathrm{d}\Omega_{kl}$ is the element of hypersolid angle for the vector $\boldsymbol{S}_{kl}$.

The integral on the hypersolid angle $\mathrm{d}\Omega_{ij}$ can be immediately performed using the technique introduced by Stanley (1969), and we get

$$Z = \sum_{\{\sigma\}} \Big[\int \prod_{(k,l)}{}' \mathrm{d}\Omega_{kl} \exp\Big(-\beta \sum_{\{k,l\}} f(J_{kl}, \boldsymbol{\sigma}_k, \boldsymbol{\sigma}_l, \boldsymbol{S}_{kl})\Big) Z_{ij}^{1-\nu/2}\, (2\pi)^{\nu/2} I_{\nu/2-1}(z_{ij})\Big], \tag{4}$$

where $z_{ij} = \beta J_{ij}(\sigma_i + \sigma_j)$, $I_{\nu/2-1}$ is the modified Bessel function and Π' means a restricted product. Following the procedure adopted before (Gonçalves 1982), we can finally write

$$Z = \sum_{\{\sigma\}} \Big[\frac{2\pi^{\nu/2}}{\Gamma(\nu/2)} \int \prod_{(k,l)}{}' \mathrm{d}\Omega_{kl} \exp\Big(-\beta \sum_{(k,l)}{}' f(J_{kl}, \boldsymbol{\sigma}_k, \boldsymbol{\sigma}_l, \boldsymbol{S}_{kl}) + \beta J'_{ij}(1 + \sigma_i\sigma_j)\Big)\Big] \tag{5}$$

where

$$J'_{ij} = \tfrac{1}{2} K_\mathrm{B} T \ln[\Gamma(\nu/2)(\beta J_{ij})^{1-\nu/2} I_{\nu/2-1}(2\beta J_{ij})]. \tag{6}$$

The first conclusion we get from equations (5) and (6) is that the effect of the decoration of the bond is to generate an effective (temperature-dependent) exchange constant between the spin pair (i, j) provided the original interaction is of the form shown in equation (1).

An immediate application of this result is to consider that each Ising spin has z near-neighbours, all the bonds are decorated and the interactions restricted to near-neighbours are given by equation (1). If N is the total number of sites we therefore write the partition function in the form

$$Z = 2\Big(\frac{\pi}{\Gamma(\nu/2)}\Big)^{Nz/2} \sum_{\{\sigma\}} \prod_{\{i<j\}} \exp[\beta J'_{ij}(1 + \sigma_i\sigma_j)] \tag{7}$$

which is the partition function of the Ising model in a d-dimensional lattice.

For the one-dimensional lattice ($z = 2$) within this approach we can recover immediately the results obtained before (Goncalves 1982). It should also be noticed that for $\nu = 1$ the procedure is entirely equivalent to the decimation transformation of renormalisation group theory. Equation (7) agrees with the one obtained by Maris and Kadanoff (1978) for the one-dimensional lattice.

2. The decomposition theorem

Let us consider now two systems A and B containing Ising and classical spins and allow the Hamiltonian $H_A(\boldsymbol{\sigma}_1, \ldots, \boldsymbol{\sigma}_k; \boldsymbol{S}_1, \ldots, \boldsymbol{S}_{k'})$ to be of the most general form. We will restrict $H_B(\boldsymbol{\sigma}_{k+1}, \ldots, \boldsymbol{\sigma}_N; \boldsymbol{S}_{k'+1}, \ldots \boldsymbol{S}_{N'})$ to a class of systems where the Ising spins appear in even products or where the classical spins generate these interactions. We will also consider that the two systems interact through a single bond connecting the Ising spins $\boldsymbol{\sigma}_k$, $\boldsymbol{\sigma}_{k+1}$ via the interaction

$$H_{AB} = -J_{AB}\boldsymbol{S}_{AB} \cdot (\boldsymbol{\sigma}_k + \boldsymbol{\sigma}_{k+1}). \tag{8}$$

We can now derive a theorem based on the one proved by Falk (1974, 1975) for systems containing Ising spins only.

Theorem. The canonical ensemble average of any function of the Ising and/or classical spins in system A only is independent of all interaction parameters contained in H_B and H_{AB}.

Proof. Let $f(\boldsymbol{\sigma}_A, \boldsymbol{S}_A) \equiv f(\boldsymbol{\sigma}_1, \ldots, \boldsymbol{\sigma}_k; \boldsymbol{S}_1, \ldots, \boldsymbol{S}_{k'})$. Then we have

$$\langle f \rangle = \frac{\sum_{\{\sigma_A, \sigma_B\}} \int \prod_{i=1}^{N'} \mathrm{d}\Omega_i \, \mathrm{d}\Omega_{AB} f(\boldsymbol{\sigma}_A, \boldsymbol{S}_A) \exp(-\beta(H_A + H_B + H_{AB})]}{\sum_{\{\sigma_A, \sigma_B\}} \int \left(\prod_{i=1}^{N'} \mathrm{d}\Omega_i\right) \mathrm{d}\Omega_{AB} \exp[-\beta(H_A + H_B + H_{AB})].}. \tag{9}$$

Since the Hamiltonians commute we can separate the exponential and perform the integral on $\mathrm{d}\Omega_{AB}$ since the spin $\boldsymbol{S}_{AB}$ appears only in the interacting Hamiltonian. Then using the result of the previous section we can write

$$\langle f(\boldsymbol{\sigma}_A, \boldsymbol{S}_A) \rangle$$
$$= \frac{\sum_{\{\sigma_A, \sigma_B\}} \int \prod_{i=1}^{N'} \mathrm{d}\Omega_i f(\boldsymbol{\sigma}_A, \boldsymbol{S}_A) \exp(-\beta H_A) \exp(-\beta H_B) \exp[\beta J'_{AB}\,(1 + \sigma_k \sigma_{k+1})]}{\sum_{\{\sigma_A, \sigma_B\}} \int \left(\prod_{i=1}^{N'} \mathrm{d}\Omega_i\right) \exp(-H_A) \exp(-\beta H_B) \exp[\beta J'_{AB}(1 + \sigma_k \sigma_{k+1})]} \tag{10}$$

where J'_{AB} is given by equation (6) by making J_{ij} equal to J_{AB}. Following closely Falk (1974, 1975) we introduce the new spin variables $t_i = \pm 1$ for $i = 1, \ldots, N$ such that

$$\sigma_j = \begin{cases} t_j & \text{for } j = 1, 2, \ldots, k \\ t_k t_j & \text{for } j = k+1, \ldots, N \end{cases} \tag{11}$$

with the inverse relations

$$t_j = \begin{cases} \sigma_j & \text{for } j = 1, \ldots, k \\ \sigma_k \sigma_j & \text{for } j = k+1, \ldots, N \end{cases} \tag{12}$$

and we can write H_A and H_B in the form

$$\begin{aligned} &H_A(\boldsymbol{\sigma}_1, \ldots, \boldsymbol{\sigma}_k; \boldsymbol{S}_1, \ldots, \boldsymbol{S}_{k'}) \equiv \bar{H}_A(\boldsymbol{t}_1, \ldots, \boldsymbol{t}_k; \boldsymbol{S}_1, \ldots, \boldsymbol{S}_{k'}) \\ &H_B(\boldsymbol{\sigma}_{k+1}, \ldots, \boldsymbol{\sigma}_N; \boldsymbol{S}_{k+1}, \ldots, \boldsymbol{S}_{N'}) \equiv \bar{H}_B(\boldsymbol{t}_{k,k+1}, \ldots, \boldsymbol{t}_{k,N}; \boldsymbol{S}_{k'+1}, \ldots, \boldsymbol{S}_{N'}) \end{aligned} \tag{13}$$

where $\boldsymbol{t}_j \equiv t_j \hat{\boldsymbol{u}}_j^{\nu}$ and $\boldsymbol{t}_{k,j} \equiv t_k t_j \hat{\boldsymbol{u}}_j^{\nu}$.

Substituting these results in equation (10) and rewriting the other Ising spin in terms of the new spin variables, we obtain

$$\langle f\rangle=\frac{\left[\sum_{\{t_A\}}\left(\int\prod_{i=1}^{k'}\mathrm{d}\Omega_i\exp(-\beta\bar{H}_A)f(t_1,\dots,t_k;\boldsymbol{S}_1,\dots,\boldsymbol{S}_{k'})\right.\right.\times\left.\left.\sum_{\{t_B\}}\int\prod_{i=k'+1}^{N'}\mathrm{d}\Omega_i\exp(-\beta\bar{H}_B)\exp[\beta J'_{AB}(1+t_{k+1})]\right)\right]}{\sum_{\{t_A\}}\left(\int\prod_{i=1}^{k'}\mathrm{d}\Omega_i\exp(-\beta\bar{H}_A)\right)\sum_{\{t_B\}}\left(\int\prod_{i=k'+1}^{N'}\mathrm{d}\Omega_i\exp(-\beta\bar{H}_B)\exp[\beta J'_{AB}(1+t_{k+1})]\right)}. \tag{14}$$

From this expression it follows that

$$\langle f\rangle=\frac{\sum_{\{t_A\}}\left(\int\prod_{i=1}^{k'}\mathrm{d}\Omega_i\exp(-\beta\bar{H}_A)f(t_1,\dots,t_k;\boldsymbol{S}_1,\dots,\boldsymbol{S}_{k'})\right)}{\sum_{\{t_A\}}\left(\int\prod_{i=1}^{k'}\mathrm{d}\Omega_i\exp(-\beta\bar{H}_A)\right)} \tag{15}$$

which proves the theorem.

An interesting application of this theorem is the calculation of the correlations between Ising spins on a Cayley tree, where the bonds are decorated by classical spins. This application is at the moment the object of study.

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